Exact Solutions for Area-based Delay Discounting Analyses

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Source Code for Simulations/Analyses (First Submitted 2019):
https://github.com/miyamot0/Exact-Solution-Model-Area

Number of text pages: 19
Number of tables: 1
Number of figures: 5

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Abstract

Novel methods are provided for calculating a model-based Area Under Curve (MB-AUC) using exact solutions in the study of delay discounting. The exact solution approach provides an AUC ratio with neither the need for numerical methods nor access to the original delay discounting data. This approach permits a calculation of MB-AUC that can be derived for both current and retrospective discounting analyses using a fitted model (e.g., *k*, *s*) and corresponding study parameters (i.e., range of delays). Simulated comparisons of numerical integration and exact solutions revealed that both approaches provided essentially identical results. The exact solution method is reviewed and demonstrated to support a comparison of results from various delay discounting analyses, including the empirical point-based Area Under the Curve (PB-AUC). Re-analyses of published study data revealed that the results of varying discounting analyses yielded similar MB-AUC ratios across groups, even as analyses and study parameters varied. The MB-AUC measure is discussed as an approach for addressing the unique challenges present when synthesizing the results of discounting studies that have used a mixture of modeled and empirical measures.

Keywords: delay discounting, decision-making, integral calculus, exact solution

Public Significance Statement: The methods included in this work provide an extension and simplification and prior analytical strategies. The approach outlined in this work supports meta-analytic syntheses of decision-making research across various domains (e.g., substance use, clinical disorders).

Introduction

Choice and decision-making are long-standing areas of interest in the applied behavioral sciences (Bickel et al., 2015; Odum, 2011). Research in this area investigates how individuals arrive at specific choices—especially when optimal choices are disregarded in favor of lesser or "risky" options. Temporal, or delay, discounting refers to how the perceived value of some item or event varies as a result of its distance from the present (Ainslie, 1975, 1974; Chung & Herrnstein, 1967). Delay discounting has been used extensively to understand "unsafe" and "impulsive" choices, and discounting has been considered a trans-disease phenotype (Mitchell, 2019). Much research is dedicated to delay discounting demonstrated by users of substances including opioids (Karakula et al., 2016), methamphetamines (Hoffman et al., 2006), cocaine (Johnson et al., 2015), cigarettes (Weidberg et al., 2017), and alcohol (Adams et al., 2017; MacKillop et al., 2010; Moore & Cusens, 2010). Research has also explored how delay discounting can affect more commonplace but potentially risky outcomes, such as automobile accidents (Freeman et al., 2017), condom use (Johnson et al., 2016), driver errors (Romanowich et al., 2020), obesity (Amlung et al., 2016), texting while driving (Hayashi et al., 2018) and gambling (Dixon et al., 2003; Madden et al., 2011).

Beyond pathological and "risky" choices, the delay discounting framework has also been used to better understand how choices in dating (Jarmolowicz et al., 2015), parenting (Call et al., 2015; Gilroy & Kaplan, 2020), dieting (Appelhans et al., 2012; Barlow et al., 2016), care for the environment (Arbuthnott, 2010; Berry et al., 2017; Carson & Roth Tran, 2009; Hardisty & Weber, 2009), credit card use (Fagerstrøm & Hantula, 2013) health management (Chapman, 2002), helping cyberbullying victims (Hayashi & Tahmasbi, 2020) and job selection can be affected by delays (Schoenfelder & Hantula, 2003). Delay is an important part of most decisions, both pathological and every day. Humans are temporal creatures, and we cannot understand human decision-making without understanding the effects and implications of delays.

Measuring and Analyzing Individual Delay Discounting

Research on delay discounting in both "risky" and every day decision-making has often been assessed using one of two experimental methods—by varying the size of rewards or by adjusting delays (Odum, 2011). In both approaches, the objective has been to determine a point where a participant is *indifferent* to the inequality between a more Smaller, Sooner Reward (SSR) and a Larger, Larger Reward (LLR; Critchfield & Kollins, 2001). This point represents an instance where some individual values a more immediate SSR equal to, or greater than, a more delayed LLR (Ainslie, 1974). Termed a *point of indifference* (POI), this value represents a point in which the presence of some delay (or probability) has resulted in a participant selecting the SSR over the LLR (Critchfield & Kollins, 2001; Odum, 2011). Both in-vivo and hypothetical forms of these tasks exist, with good support that hypothetical decision-making tasks correlate with their real-world equivalents (Lagorio & Madden, 2005; Madden et al., 2003, 2004). These tasks have also been found to have good test-retest reliability (Smith & Hantula, 2008). In sum, discounting research has largely focused on these SSR/LLR comparisons and the overarching methodology for performing them has remained largely consistent since the adjusting procedure was proposed (Mazur, 1987).

Although the methods for obtaining POIs have remained consistent, many approaches have been put forward to quantify and compare these data (Doyle, 2013; McKerchar et al., 2009). As indicated in reviews, the range of methods used to analyze individual choices is considerable (Amlung et al., 2017; Barlow et al., 2016, 2017; MacKillop et al., 2011; Reynolds, 2006). For example, MacKillop et al. (2011) surveyed the literature on addictive behavior and delayed reward discounting (DRD; i.e., temporal discounting) and found good support for using a variety of discounting methods to discriminate between clinical groups and controls. Of the 64 studies included in this review, 70% of studies fitted the Hyperbolic model (Mazur, 1987), 14% used the point-based Area Under the Curve procedure (PB-AUC; Myerson, Green, & Warusawitharana, 2001), and the remainder used some other empirical measure based on the data (e.g., Impulsive Choice Ratio). Although the numerous approaches used in these studies were effective in discriminating clinical groups from controls, the varying approaches complicate the overall synthesis of these findings. As noted by MacKillop and colleagues, "It is plausible and probable that such assessment parameters are differentially well suited for examining DRD (*delayed reward discounting*) and addictive behavior and a profitable target for future studies would be clarifying the more sensitive measures (p. 316)."

Throughout the temporal discounting literature, discounting processes have often been quantified using a mix of modeled (i.e., statistical) and empirical (i.e., un-modeled) approaches. Within the methods using statistical modeling, researchers have often employed model fitting and reported individual discounting as fitted parameters (e.g., Mazur's Hyperbolic *k*). These values are generally reported normally or in some transformed state (e.g., logarithmic, natural logarithm, square root). Frequent use of these measures is understandable, as model parameters often have good statistical properties following some method of transformation (Mitchell et al., 2015; Yoon et al., 2017). Alternatively, researchers can use the empirical PB-AUC method introduced in Myerson et al. (2001) to geometrically represent POIs as the area beneath a trapezoidal interpolation of a series of points comprising a delay curve. In both modeled and empirical approaches, these methods have been used to represent how *steeply* or *rapidly* individuals advance from optimal decisions to more "impulsive" ones. That is, larger *k* values

(Mazur, 1987) and smaller PB-AUC measurements (Myerson et al., 2001) are indicative of greater discounting. While both methods can be used to compare degrees of discounting, it warrants noting that each approach quantifies discounting differently and that these differences limit the degree to which they can be compared directly.

As a related but distinct set of methods, modeled discounting processes are also analyzed in ways that facilitate generalized comparisons across models. First, discounting can be represented using some shared ordinate of interest, such as the midpoint between a starting value (i.e., the full value of commodity) and zero on the y-axis (i.e., 50% decay). The Effective Delay 50 (ED50; Yoon & Higgins, 2008) can be used across instances of modeled discounting because it solves for some value on the x-axis (i.e., delay) where a discounting process arrives at 50% of its original value. This approach allows for comparisons when models vary and can be used as a general index of how rapidly the value of some good was discounted. Using the ED50, smaller values would indicate that discounting occurred more rapidly while larger values would indicate that discounting took place less rapidly.

Second, discounting processes can also be interpreted based on their form between two bounds of interest (i.e., first and final delay). Rather than solving for a single point, such as with ED50, the emphasis here is on the full range changes between two points in time. Using integration, a fitted discounting process can be represented as the amount of area beneath it between two points in time (i.e., the first and final delay). Whereas the empirical PB-AUC approach summarizes the area between linearly interpolated data points, the model-based AUC (MB-AUC) approach represents the area beneath an instance of modeled discounting (Gilroy & Hantula, 2018). In the MB-AUC approach, integration is performed upon the fitted discounting curve, rather than the data itself. Similar to the ED50, the MB-AUC measure can also be applied and interpreted across various discounting models. The MB-AUC process can also be scaled in multiple dimensions, consistent with the suggestions in Borges et al. (2016). In sum, the MB-AUC and PB-AUC measures are highly similar in their interpretation and MB-AUC provides a good basis for making comparisons between modeled and PB-AUC.

Comparing Discounting Measures

Relationships between modeled, empirical, and generalized measures of discounting have historically been complex (Mitchell et al., 2015; Yoon et al., 2017). For modeled measures, varying model structures and parameters make comparisons challenging and/or inappropriate when the models applied are not identical. That is, individual rate parameters (e.g., *k*, *s*) from differing models cannot be equated to one another due to varying model structures, and incorporating additional parameters further complicates such comparisons. Similarly, empirical measures such as PB-AUC are difficult to compare when the data included vary in number and the range of delays sampled. For example, PB-AUC constructed using delays between 1 and 2160 days cannot be directly compared to a PB-AUC measure constructed with delays between 1 and 4320 days. In such cases, PB-AUC would very likely be much lower for the 4320-day delay series because it would account for *assumed* higher levels of discounting at greater delays that would not have been represented in the 2160-delay series.

Although conceptually neutral, the PB-AUC calculation is bound by the design of the experiment and/or the data supplied[1]. As illustrated in Figure 1, the PB-AUC ratio is substantially influenced by the range of delays. That is, broader ranges of delays are more prone to deflating PB-AUC. For example, the discounting series depicted in the top panel of Figure 1

[1] We make note that PB-AUC is occasionally comparable to other PB-AUC with differing ranges when certain data can be *removed,* often the first or final points, and the remaining are matched in count and range. However, the removal of suitable and appropriate data may unintentionally truncate variance of interest to the researcher and this is generally undesirable.

portrays a series wherein no further discounting is observed following a delay of 3650 days (10 years). Discounting is assumed to be unchanged at further delays of 20 and 40 years. Despite no further observed discounting past a delay of 10 years, the proportion of the area is increasingly deflated as the range of delays increases. The result of the increasing delay range is a proportion in which the area under the curve continues to grow less rapidly while the maximum area possible increases linearly. As such, larger delay spans will nearly always result in smaller AUC ratios (see the middle and bottom panels of Figure 1). Because of this limitation, one PB-AUC may be smaller than another PB-AUC not due to increased rates of discounting, but rather the design of the experiment and the range of delays from which that quantity was derived.

Given that modeled and empirical discounting methods represent discounting differently, both approaches must be interpreted separately. This makes good sense, as each metric represents discounting differently (e.g., rate, area). However, the characteristics of the newly developed MB-AUC offer a new avenue for representing modeled discounting in novel ways. First, as a measure of area, MB-AUC represents the area under a modeled discounting process between two points in time (i.e., from 1 day to 5 years). It is an area ratio much like PB-AUC but derived from a fitted discounting process rather than points of data. Unlike PB-AUC, MB-AUC is free from the constraints of the data and can be calculated with any two delays of interest—it is not bound to the data supplied as in PB-AUC. That is, it can be adapted as necessary depending on the delays of interest. In all other ways, it is essentially the same manner of area calculation so long as the range of delays is kept the same.

Second, MB-AUC is a proportion of area beneath a discounting process between *any* two bounds of interest. This contrasts with PB-AUC, where the area can only be calculated between delays that are directly measured. The MB-AUC method allows for the calculation of area between *any* two delays (e.g., first delay, last delay, 10 years). For example, a researcher may have fitted a model to several POIs between delays of 1 day and 8640 days but still perform use the MB-AUC method to calculate an index of discounting between delays of 1 day and 4320 days—or nearly any other unmeasured delay. Researchers do not need to remove data from consideration to adjust the range of delays represented in the analysis, as the model itself is what is integrated. This quality of MB-AUC is desirable because it can be adapted to situations when other measures, such as PB-AUC, are bound to specific delay values and cannot be directly compared beyond them. For example, a researcher may use MB-AUC as an intermediary when comparing a well-performing instance of modeled discounting (e.g., Mazur's Hyperbolic *k*) to an instance of empirical discounting (i.e., PB-AUC) and this comparison can be performed even when the original delay points differ[2]. This can be done by applying the bounds from the PB-AUC calculation to the modeled discounting process, using MB-AUC. In doing so, MB-AUC shares the same domain as the PB-AUC analysis and the area represented in both analyses reflects the same maximum possible area. That is, differences between the two analyses would be from the degree of discounting observed and free from the potential error resulting from comparing AUC with differing delay ranges.

All commonly used methods for calculating POI can distinguish between clinical and non-clinical samples. However, as research in delay discounting further includes non-clinical samples and questions about more quotidian behaviors that may not yield the same stark differences, more sensitive indices are needed, as MacKillop et al. (2011) assert. Further, there is growing interest in using meta-analytic and other quantitative techniques to summarize

[2] We make a note here that PB-AUC and MB-AUC are similar when the modeled discounting process represents the observed data well. The MB-AUC metric is an integration of the fitted model and the result is a measure as representative of the data as the model supplied.

behavioral literature (Dowdy et al., 2021, 2022). Such analyses require dependent variables on similar scales that can be readily compared, which is not the case for most discounting measures. Hence, the absence of a comparable metric may represent a significant barrier to advances in delay discounting research.

Exact Solutions for Model-based Area

The original MB-AUC method put forward by Gilroy et al. (2018) used discounting model selection and numerical integration to determine an area ratio from delay discounting data. Numerical integration, the original process for calculating MB-AUC, is an iterative process whereby a very large number of area segments are constructed to numerically approximate the area beneath a discounting function. Through a process of constructing many very small area segments, the degree of error decreases to a point that a final area calculation is approximate to the true, or exact, solution[3]. Numerical integration can be performed by many statistical packages, including the Discounting Model Selector computer program (DMS; Gilroy, Franck, & Hantula, 2017; Gilroy & Hantula, 2018) and other statistical packages (e.g., Gilroy, 2022). Solving for the area beneath a discounting process can be done in other ways as well even without access to source data. Although MB-AUC can be determined using numerical integration, it can also be determined via exact solutions.

In contrast to numerical integration, the area beneath a fitted function between two bounds can be solved analytically using integral calculus. Such methods are regularly used in the application of Bayesian methods, such as in the determination of probability density. This approach, the exact solution for MB-AUC, requires only the model, fitted parameter(s), and two

[3] When we speak of exact solution, we are referring to an analytic or symbolic solution. This is in contrast to a numerical solution, wherein a solution is iteratively updated in efforts to decrease error to very low levels, and thus, approach the analytic solution.

bounds of interest to perform. Access to the source data is not required because the necessary information is contained in the fitted model parameters and is bounded by the delays of interest. In doing so, the solution does not need to *approximate* the area numerically because it can be solved symbolically. Exact solutions for calculating MB-AUC exist for the Exponential (Samuelson, 1937), Hyperbolic (Mazur, 1987), Quasi-Hyperbolic (Laibson, 1997), Green-Myerson (Green & Myerson, 2004), and Rachlin (Rachlin, 2006) discounting models and these calculations are provided in the Appendix of this paper. Using these solutions, which are generally calculable using basic spreadsheet software[4], MB-AUC can be derived using only the fitted parameters from discounting analyses. As such, exact solutions for calculating MB-AUC allow researchers to easily construct an area-based representation of modeled discounting across varying models and delays using only the model, the fitted parameter(s), and two delay points of interest. These data are typically reported in peer-reviewed works, so researchers can also use this approach to facilitated comparisons between experiments using this newer method.

The process of solving for the exact solution MB-AUC requires the maximum value of the commodity (A), the first delay of interest (T_1), the final delay of interest (T_2), and the respective model and fitted parameter(s). Using these values, the area between T_1 and T_2 can be solved using the integral between T_1 and T_2. By subtracting the initial area from the final area, the resulting difference represents the total area under the curve between T_1 and T_2. Once divided by the maximum possible area (i.e., the span between T_1 and T_2 multiplied by A), the result is the ratio of the area beneath the discounting function to the maximum possible area—consistent with the interpretation of PB-AUC and MB-AUC. Exact solutions for each of the models described above are provided in the Appendix and scripts for demonstrating how to apply these solutions

[4] Although the exact solutions provided are generally calculable in basic software, models such as the Rachlin hyperboloid model require specialized methods that are likely limited to proper statistical and mathematical tools.

have been provided for the R, Python, and Matlab software suites for convenience and replication.

Although methods such as numerical integration already exist for determining MB-AUC, exact solutions for MB-AUC add to the tools available to discounting researchers in several ways. First and foremost, these solutions allow researchers to easily convert a fitted discounting function directly into an area proportion. That is, researchers can easily convert existing discounting results into MB-AUC even when discounting models vary, and it can be done without access to the original data. This allows for re-analyses of existing study results, where the source data may not be available but modeled and empirical measures are reported. Second, the exact solution MB-AUC allows researchers to make a comparison of modeled and empirical discounting studies. For example, consider the following situation; a researcher may wish to compare the results from three similar studies that indexed individual discounting differently. For the sake of convenience, these will be referred to as Studies A, B, and C. In Study A, the PB-AUC method was used on data with delay points of 7, 14, 30, 183, 365, and 1825 days and this produced a group mean PB-AUC of 0.677. Study B modeled individual discounting using Mazur's Hyperbolic model to data with delay points of 1, 7, 14, 30, 183, 365, 1825, and 9125 days. The k value fitted to the group data was 0.002357. Lastly, Study C modeled individual discounting using the Green-Myerson Hyperboloid model to data with delays of 1, 7, 14, 30, 183, and 365 days. In fitting this model to the aggregated data this model resulted in the best fit k of 0.043864 and s of 0.219506. For convenience, these data and descriptions are also provided in Table 1 and individual discounting processes are illustrated in Figure 2.

In comparing the three hypothetical studies, Study A cannot be equated to B or C because PB-AUC cannot be directly compared to modeled decision-making. Studies B and C cannot be directly compared because the models themselves differ in both their structure and number of parameters. Although ED50 can indeed be used to compare B and C, this would not allow for a comparison of Studies B and C to A. Numerical integration MB-AUC for studies B and C allows for a comparison of area ratios; however, these would be prone to the error introduced when the delay ranges differ between calculations. Fortunately, the exact solution MB-AUC methods can be used with Study B and Study C by adopting the minimum and maximum delay values of Study A. In this way, the area proportions for these studies are made comparable so long as some process of discounting is present. After solving for the exact solution of modeled studies, the PB-AUC from Study A was 0.6771 while the MB-AUC from Study B and Study C were 0.3854 and 0.4764, respectively.

A Novel Synthesis of Delay Discounting Results

The ability to flexibly represent discounting processes as either fitted functions or area ratios offers several new possibilities to applied researchers. First, a robust and easily-adapted area measure allows for comparisons that have historically been incomparable. That is, exact solutions allow applied researchers to compare measures of decision-making even when analytic strategies vary. Further, these methods also facilitate a more robust synthesis of existing decision-making research. For example, historical studies can be revisited and directly compared to more recent results. This is especially desirable as more advanced modeling practices (e.g., Rachlin and Green-Myerson models) are increasingly enlisted over more traditional approaches (e.g., Samuelson and Mazur models). Second, exact solutions provide a straightforward way for researchers to present their results in multiple ways—which is consistent with recommended practices (Mitchell et al., 2015). For example, applied researchers can elect to model individual and group decision-making and then perform exact solutions to calculate a supplemental measure of AUC based on their statistical modeling.

To evaluate the utility of the exact solutions reviewed in this work, simulations and demonstrations with existing data were necessary to explore the accuracy and reliability of this novel approach. Simulations were performed to validate the applicability of solutions for each of the given discounting models with respect to the earlier numerical integration methods. Individual computer simulations were developed to evaluate the accuracy and reliability of exact solution methods across the several solutions provided and data were extracted from peer-reviewed studies to compare the results from modeled and empirical AUC calculations. All source code and data necessary to recreate the data and analyses of this work have been made publicly available and further information is available in the Appendix of this work. This study asked the following questions: 1) to what degree do the results from exact solution MB-AUC correspond with results from numerical integration MB-AUC overall, 2) to what degree do individual solutions related to the results from numerical integration MB-AUC, and 3) does a comparison of PB-AUC and MB-AUC derived from existing peer-reviewed studies conform to existing predictions regarding clinical and non-clinical populations?

Methods

Simulation Study

Data were simulated to compare results from numerical integration MB-AUC to results derived from exact solution MB-AUC across various model candidates. Simulations were individually constructed for each of the Exponential, Hyperbolic, Quasi-Hyperbolic, Rachlin, and Green and Myerson models (n = 10,000 x 5 = 50,000). Individual simulations were identical to that of Franck et al. (2015) and were designed to replicate the numerical integration MB-AUC results used in Gilroy et al. (2018). Simulated series included delays of 1, 7 (1 week), 14 (2 weeks), 30 (1 month), 183 (6 months), 365 (one year), 1825 (5 years), and 9125 days (25 years). Simulated series were based on parameters derived from an existing hypothetical monetary delay discounting task (see Franck et al., 2015). Both simulated data as well as the resources necessary to replicate them have been openly shared and instructions for obtaining these resources are provided in the Appendix.

Numerical Integration Area

Numerical integration MB-AUC was performed using the DMS computer software. The DMS computer uses a discounting model selection procedure based on approximate Bayesian model selection (see Franck et al., 2015) to identify the most probable discounting model for an instance of discounting before performing numerical integration. Numerical integration was applied to fitted discounting functions using the *autogkintegrate* method in the ALGLIB programming library (Bochkanov & Bystritsky, 1999). The first and final delays specified the upper and lower bounds of the domain, respectively, and the most probable discounting model was selected to be function integrated. Once this area was solved, this value was divided by the maximum area possible, yielding a ratio of area under the curve to total area.

Exact Solution Area

In contrast to numerical integration MB-AUC, exact solutions did not require an iterative process. Exact solutions were used following discounting model selection, using delay extremes as T_1 and T_2, respectively, and a maximum value of one. The exact solution method and model parameters used were based on the results of nonlinear model-fitting and subsequent discounting model selection. Although nearly most solutions did not require specialized methods to perform, the Rachlin solution required the use of a hypergeometric function. This function, ${}_2F_1(a; b; c; z)$, is used in solving second-order ordinary differential equations. The hypergeometric function was solved in the Python computer programming language using the *hyp2f1* method contained in the SciPy scientific computing library (Jones et al., 2001), though similar implementations exist in other programming languages. Implementations for each of the models included are provided in source code form within the Appendix of this paper in several programming languages (i.e., R, Python, and Matlab).

Extracted Study Results

A total of five studies comparing the rates of discounting for monetary outcomes in smokers and non-smokers were re-analyzed using a combination of PB-AUC and exact solution MB-AUC. Individual studies included magnitudes that ranged from $10 to $1,000, though individual studies were scaled with respect to their magnitude of rewards. Within the five studies included in this comparison, one study used the PB-AUC measure to compare rates of discounting (Reynolds et al., 2007), one used Myerson & Green's Hyperboloid model (Friedel et al., 2014), and three studies used Mazur's Hyperbolic model (Bickel et al., 1999; Mitchell, 1999; Reynolds et al., 2004). Several studies included additional groups (e.g., ex-smokers), though these groups were not included as part of the smoker vs non-smoker comparisons. In re-analyzed studies using exact solution MB-AUC, the models and median fitted parameter(s) were used to construct a group-level MB-AUC for each study. Exact solutions were calculated using the range of delays targeted in the included PB-AUC study. The PB-AUC study reported PB-AUC using the median indifference point at each of the delays sampled. Although the modeled studies sampled delays that included 1 day, 1 week, 2 weeks, 1 month, 6 months, 5 years, 10 years, and 25 years, the PB-AUC study sampled delays of 1, 2, 30, 180, and 365 days. To address the limitations of the PB-AUC measure, the exact solution MB-AUC evaluated the domain of changes between 1 and 365 days in all modeled studies.

Results

Simulation Results

The overall relationship between the two MB-AUC methods is illustrated in Figure 3. Wilcoxon Rank Sum Tests indicated that there was not a significant difference between the two methods, $W = 1.2497\text{x}10^9$, $p = .95$, $rho = .99$. Formal tests of equivalence were considered unnecessary because the results from each method generally matched up to the fifth decimal place. At the individual model-level, there were no significant differences found at the individual model level for the Noise, $W = 6.728 \times 10^5$, $p = .99$, $rho = 1$, Exponential, $W = 3.9214 \times 10^7$, $p = 0.99$, $rho = 1$, Hyperbolic, $W = 6.0709 \times 10^7$, $p = 1$, $rho = 1$, Quasi-Hyperbolic, $W = 6.0822 \times 10^7$, $p = .77$, $rho = 0.99$, Green & Myerson, $W = 1.8648 \times 10^7$, $p = 1$, $rho = 1$, and Rachlin models, $W = 6.9809 \times 10^7$, $p = 0.99$, $rho = 1$. The relationships between both methods, at each model level, are illustrated in Figure 4. The results of these comparisons indicated that all exact solutions for MB-AUC provided results that were essentially identical to those from numerical integration methods up to the fifth decimal place.

Extracted Study Results

The results from four modeled discounting studies converted to MB-AUC and one study reported using PB-AUC are illustrated in Figure 5. Cast into the same range of delays and metric of discounting, these results differentiated clinical populations from non-smoking controls within individual studies. A similar trend was also observed overall. The synthesis of these individual results revealed that clinical populations demonstrated overall higher levels of discounting ($M = 0.195$, $SD = 0.106$) than controls ($M = 0.385$, $SD = 0.121$) when interpreted in aggregate.

Discussion

Despite having generally consistent and established methods for assessing intertemporal choice, the analytical strategies used to quantify these data continue to remain largely divergent between modeled and empirical approaches. This is limiting in many respects, as the results from one analytic strategy are not readily comparable to another. The result of this has been a relative paucity of meta-analytic work synthesizing this literature. The purpose of this study was to evaluate a method that could serve as an intermediary between modeled and empirical analyses and the following questions were posed: 1) to what degree do the results from exact solution MB-AUC correspond with results from numerical integration MB-AUC overall, 2) to what degree do individual solutions related to the results from numerical integration MB-AUC, and 3) does a comparison of PB-AUC and MB-AUC derived from existing peer-reviewed studies conform to existing predictions regarding clinical and non-clinical populations? Based on the results of this study, both forms of MB-AUC provided consistent area-based representations of modeled decision-making. However, the exact solution method for determining MB-AUC did so without requiring the data from which the model was generated. This indicates that the solutions presented now offer a much easier method for translating modeling fits into area metrics. Additionally, the findings from five studies that used different methods to compare delay discounting between smokers and non-smoking controls yielded consistent area differences when modeled discounting was represented as MB-AUC.

The exact solution methods are a substantial expansion of numerical integration MB-AUC for several reasons. First and foremost, exact solution allows for an easy translation of fitted discounting processes into area measures. For example, the most commonly-used models of discounting can be converted to MB-AUC using nothing more than spreadsheet software or a basic scientific calculator (e.g., Hyperbolic model). Given the relative ease of these calculations, exact solution methods may serve as an effective supplement even in studies where modeled discounting is the primary analytical plan but additional metrics help communicate the discounting process. In doing so, this newer approach may assist researchers in reporting the results of their studies using multiple measures (Mitchell et al., 2015).

Second, exact solution provides an unprecedented opportunity to compare modeled discounting processes and empirical area estimates, namely PB-AUC. Following the first point, exact solution provides a mathematically-sound method for converting a process of discounting directly into an area proportion. In doing so, MB-AUC can serve as an intermediary between two historically dichotomous analytical strategies. The knowledge of an exact solution for model area may serve to better clarify the historically complex relationships observed between individual rate parameters and empirical measures of discounting, such as PB-AUC (Mitchell et al., 2015; Yoon et al., 2017).

Third, exact solution opens new avenues for synthesizing the current research literature. That is, exact solution MB-AUC can be performed using the results from published study data, thus access to the raw data is not necessary. This is especially important, as many studies rarely provide the raw data necessary to determine MB-AUC using a process of model fitting and subsequent numerical integration. For example, revisiting MacKillop et al. (2011), exact solution allows for a quantitative comparison across various choice measures using a unifying metric of AUC. MacKillop et al. could use exact solution to calculate MB-AUC along with PB-AUC and present up to 84% of their reviewed studies using an AUC-based measure (i.e., 70% Hyperbolic model + 14% PB-AUC). Through the use of a shared measure, future systematic review and meta-analysis may be able to better reveal differences that would have otherwise gone unnoticed if studies could not have been directly compared. For example, in the five studies that were included in the comparison of PB-AUC and MB-AUC, the smoking group demonstrated higher levels of discounting than their control group, but this difference was not as large as observed in other studies comparing discounting in smokers and non-smokers. That is, the smokers in Mitchell et al. (1999) demonstrated rates of discounting similar to non-smokers in other, similar studies. Through the use of a common metric, such as MB-AUC, future research may be better equipped to detect differences between individual studies—even when the specific metric differs.

Limitations of MB-AUC without Model Selection Procedures

Although exact solutions for delay discounting models present applied researchers with new opportunities for analyzing and synthesizing results from delay discounting analyses, it requires noting that converting a fitted model to area is not a panacea for the challenges associated with poorly fitting models. Just as one would inspect the quality of data before performing model fitting, it is necessary to inspect the quality of the model fit before converting to MB-AUC. This is paramount, as the MB-AUC of a poorly fitting model will be no more useful than the poorly performing model. For this reason, the MB-AUC measure was first introduced as a measure that followed discounting model selection procedures (Franck et al., 2015). In this way, the best-performing model was selected to be the one from which MB-AUC was determined. As such, it is recommended that researchers with access to raw discounting data perform model selection procedures so that a range of possible discounting models can be considered. In doing so, potential differences between PB-AUC and MB-AUC due to poor model fit can be minimized.

Further study is necessary to better understand the relationship between MB-AUC and PB-AUC. Earlier research has described PB-AUC as a sort of saturated segmented model and found that PB-AUC and MB-AUC provide very similar results when MB-AUC is derived from a well-performing model (Gilroy & Hantula, 2018). However, given the unique nature of both calculations, more study is necessary to better understand how these two area approximations represent discounting. That is, further research is necessary to understand how PB-AUC and MB-AUC are related when few data points are available, when few model candidates (or just a single model) are considered, and when the ranges of delays sampled vary significantly.

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Appendix

Exact Solutions for Model Area in Common Delay Discounting Models

$$Exponential\ Model = Ae^{-dT}$$

$$A = 100\ (Max\ Value)$$

$$T_1 = 1\ (Day)$$

$$T_2 = 9125\ (Days)$$

$$d = 0.0015$$

$$Area\ at\ T_1 = -\frac{A * e^{(-dT_1)}}{d} = -65774.1470$$

$$Area\ at\ T_2 = -\frac{A * e^{(-dT_2)}}{d} = -0.0635$$

$$Maximum\ Area = (T_2 - T_1) * A = 912400$$

$$Exact\ Area = Area\ at\ T_2 - Area\ at\ T_1 = 65774.0835$$

$$Exact\ AUC = \frac{Exact\ Area}{Maximum\ Area} = \frac{65774.0835}{912400} = 0.07209$$

$$Hyperbolic\ Model = \frac{A}{1+kT}$$
$$A = 100\ (Max\ Value)$$
$$T_1 = 1\ (Day)$$
$$T_2 = 9125\ (Days)$$
$$k = 0.0023$$
$$Area\ at\ T_1 = \frac{A * ln(1+kT_1)}{k} = 99.8865$$
$$Area\ at\ T_2 = \frac{A * ln(1+kT_2)}{k} = 135444.0412$$
$$Maximum\ Area = (T_2 - T_1) * A = 912400$$
$$Exact\ Area = Area\ at\ T_2 - Area\ at\ T_1 = 135344.1548$$
$$Exact\ AUC = \frac{Exact\ Area}{Maximum\ Area} = \frac{135344.1548}{912400} = 0.1483$$

$$Quasi\text{-}Hyperbolic\ Model = Abe^{-dT}$$
$$A = 100\ (Max\ Value)$$
$$T_1 = 1\ (Day)$$
$$T_2 = 9125\ (Days)$$
$$b = 0.9868$$
$$d = 0.0014$$
$$Area\ at\ T_1 = \frac{-Abe^{(-dT_1)}}{d} = -66945.0663$$
$$Area\ at\ T_2 = \frac{-Abe^{(-dT_2)}}{d} = -0.0983$$
$$Maximum\ Area = (T_2 - T_1) * A = 912400$$
$$Exact\ Area = Area\ at\ T_2 - Area\ at\ T_1 = 66944.96798$$
$$Exact\ AUC = \frac{Exact\ Area}{Maximum\ Area} = \frac{66944.9679}{912400} = 0.0733$$

$$Green\text{-}Myerson\ Model = \frac{A}{(1+kT)^s}$$
$$A = 100\ (Max\ Value)$$
$$T_1 = 1\ (Day)$$
$$T_2 = 9125\ (Days)$$
$$k = 0.0000021$$
$$s = 716.712$$
$$Area\ at\ T_1 = \frac{A*(kT_1+1)^{1-s}}{k*(1-s)} = -65912.1096$$
$$Area\ at\ T_2 = \frac{A*(kT_2+1)^{1-s}}{k*(1-s)} = -0.0747$$
$$Maximum\ Area = (T_2 - T_1)*A = 912400$$
$$Exact\ Area = Area\ at\ T_2 - Area\ at\ T_1 = 65912.0348$$
$$Exact\ AUC = \frac{Exact\ Area}{Maximum\ Area} = \frac{65912.0348}{912400} = 0.07224$$

$$Rachlin\ Model = \frac{A}{1+(kT)^s}$$
$$A = 100\ (Max\ Value)$$
$$T_1 = 1\ (Day)$$
$$T_2 = 9125\ (Days)$$
$$k = 0.00002$$
$$s = 1.8047$$
$$Area\ at\ T_1 = A*T_1*{}_2F_1(1,\frac{1}{s};1+\frac{1}{s};-kT_1^s) = 99.9993$$
$$Area\ at\ T_2 = A*T_2*{}_2F_1(1,\frac{1}{s};1+\frac{1}{s};-kT_2^s) = 67384.6027$$
$$Maximum\ Area = (T_2 - T_1)*A = 912400$$
$$Exact\ Area = Area\ at\ T_2 - Area\ at\ T_1 = 67284.6034$$
$$Exact\ AUC = \frac{Exact\ Area}{Maximum\ Area} = \frac{67284.6034}{912400} = 0.0737$$

Note: Solving for the exact area in the case of the Rachlin method requires the use of an ordinary hypergeometric function. This operation can be performed using any of numerous free and open-source statistical and mathematical tools, such as the R or Python programming languages.

ED50 Solutions for Common Discounting Models

$$Exponential\ ED50 = \frac{ln(2)}{k}$$

$$Hyperbolic\ ED50 = \frac{1}{k}$$

$$Quasi\text{-}Hyperbolic\ ED50 = log_{1-d}(\frac{1}{2*b})$$

$$Green\text{-}Myerson\ ED50 = \frac{2^{1/s}-1}{k}$$

$$Rachlin\ ED50 = (\frac{1}{k})^{1/s}$$

Note: Solutions listed here have been published previously in Franck et al. (2015). These values represent the ED50 without any method of transformation yet applied.

Scripts and Program Code for Performing Analyses

Scripts for performing the exact solution MB-AUC have been prepared for use in the R, Python, and Matlab programming languages. Scripts have been written to demonstrate the functionality and replicability of these solutions across several free and commercial platforms. All scripts have been open-sourced under the General Public License, Version 3.0, and can be downloaded from the corresponding author's GitHub page at the following location: https://github.com/miyamot0/Exact-Solution-Model-Area

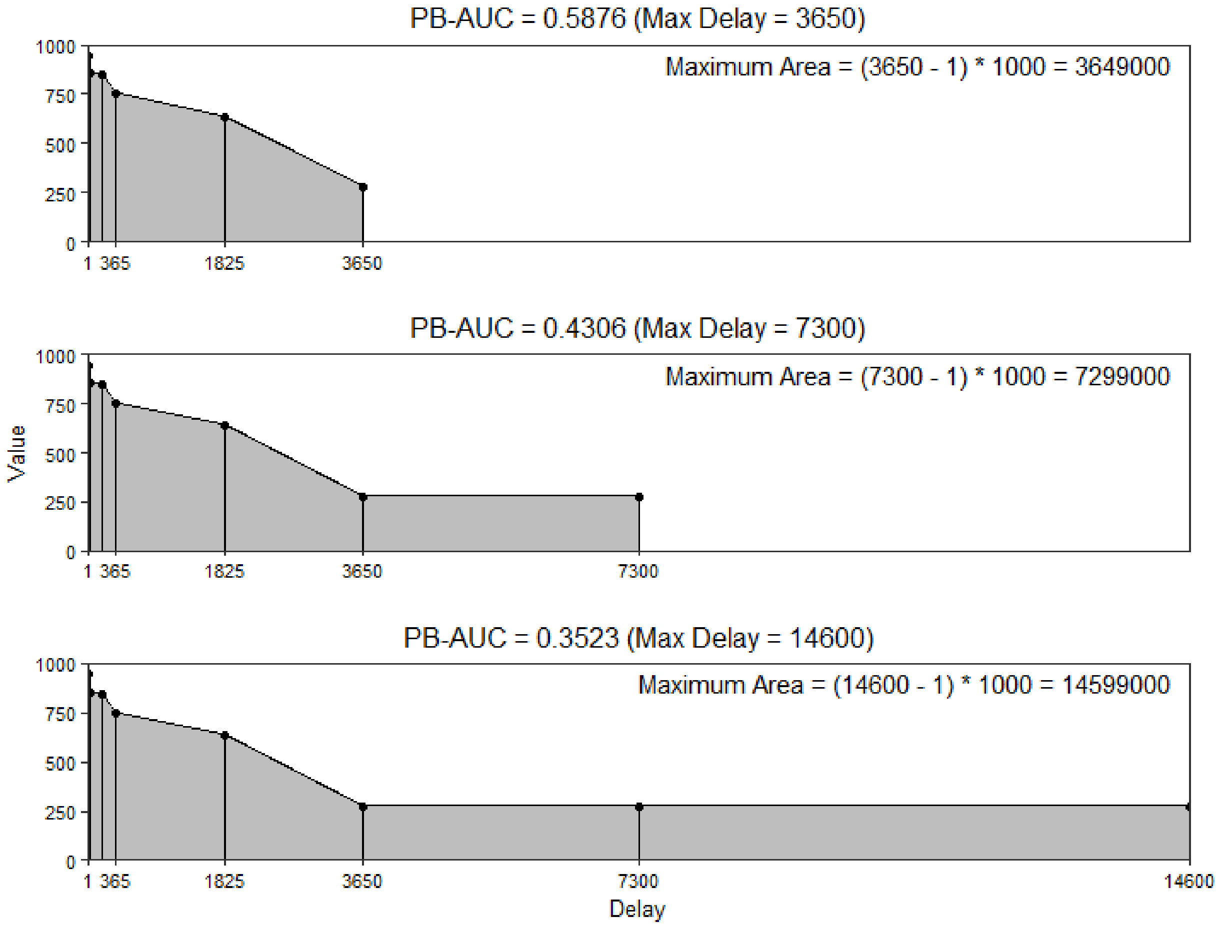
Table 1

Example of comparative MB-AUC across various delay discounting methods

Case Example	Method	Parameters	Area Measure
Study A	Point-based AUC	---	0.6771 (PB-AUC)
Study B	Mazur Model Fitting	*k*: 0.002357	0.3854 (MB-AUC)
Study C	Myerson-Green Model Fitting	*k*: 0.043864 *s*: 0.219506	0.4764 (MB-AUC)

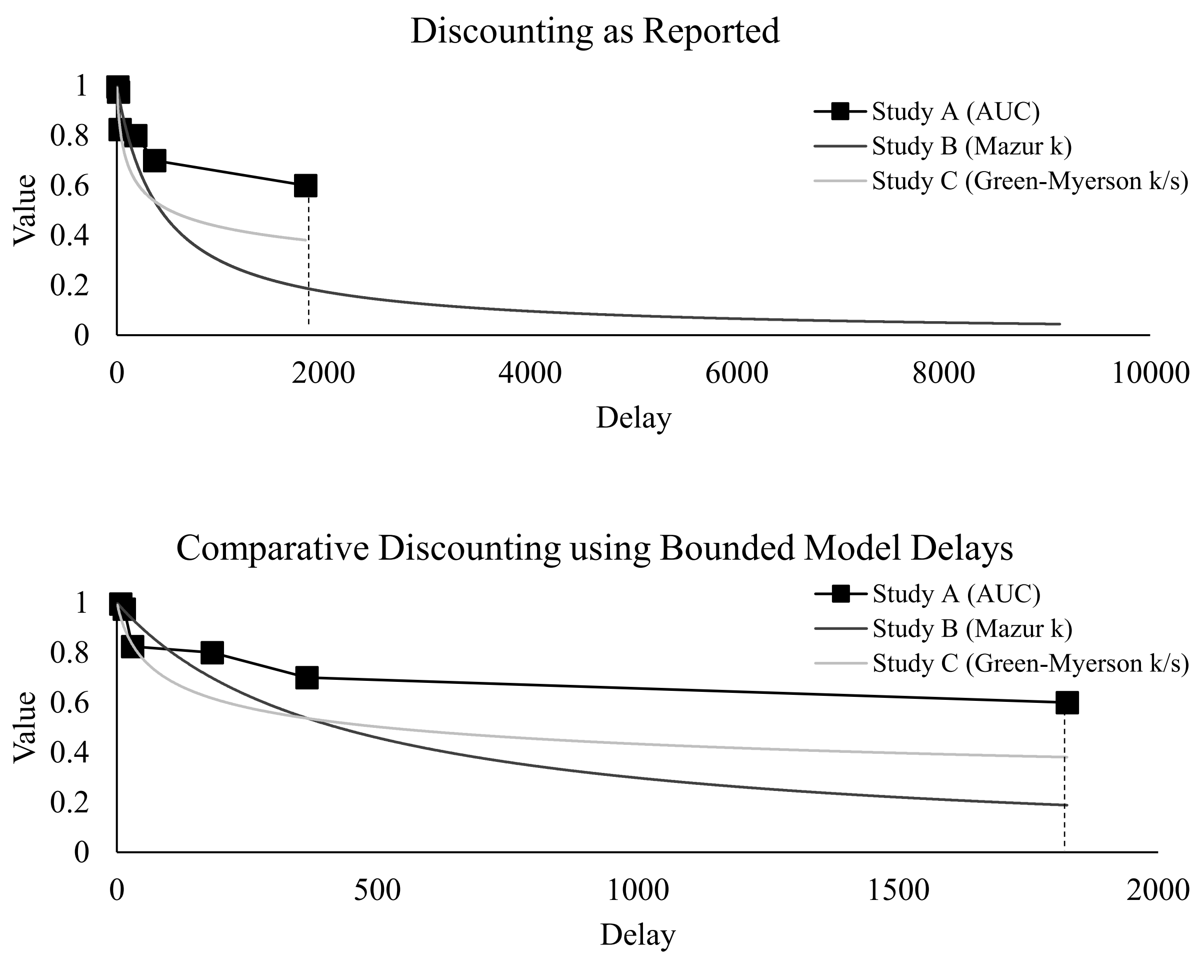
This table provides the results from the exact solution MB-AUC. The MB-AUC area measure is derived from two delays of interest ($T_1 = 7$; $T_2 = 1825$), the maximum value of the commodity (A), and the fitted model parameters.

Figure 1. Point-based area calculations as a function of delays



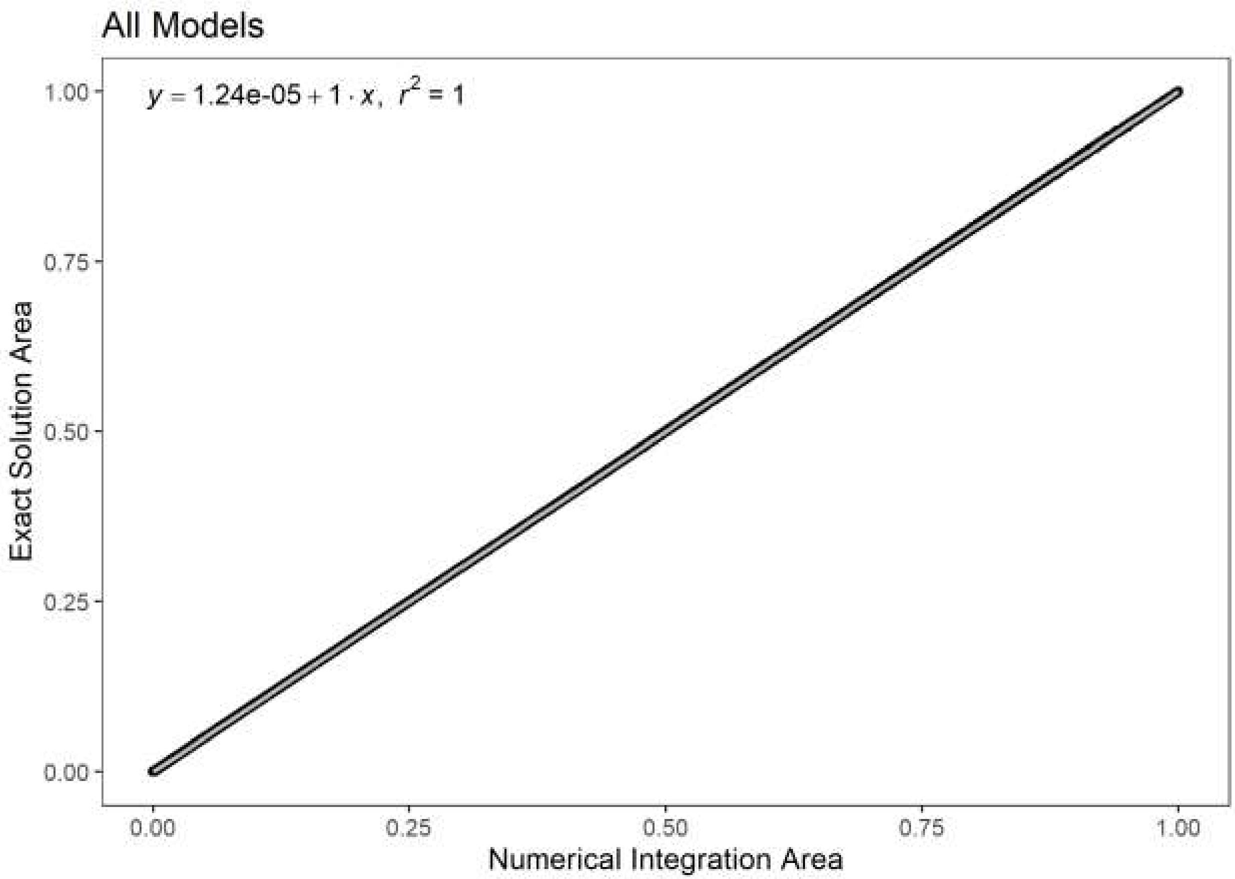
This figure illustrates how area proportions are affected by both the selection of delay values as well as the degree of discounting. That is, the area proportions become increasingly smaller due to the size and range of the maximum delay points, rather than as a function of the degree of discounting.

Figure 2. Comparative discounting between studies



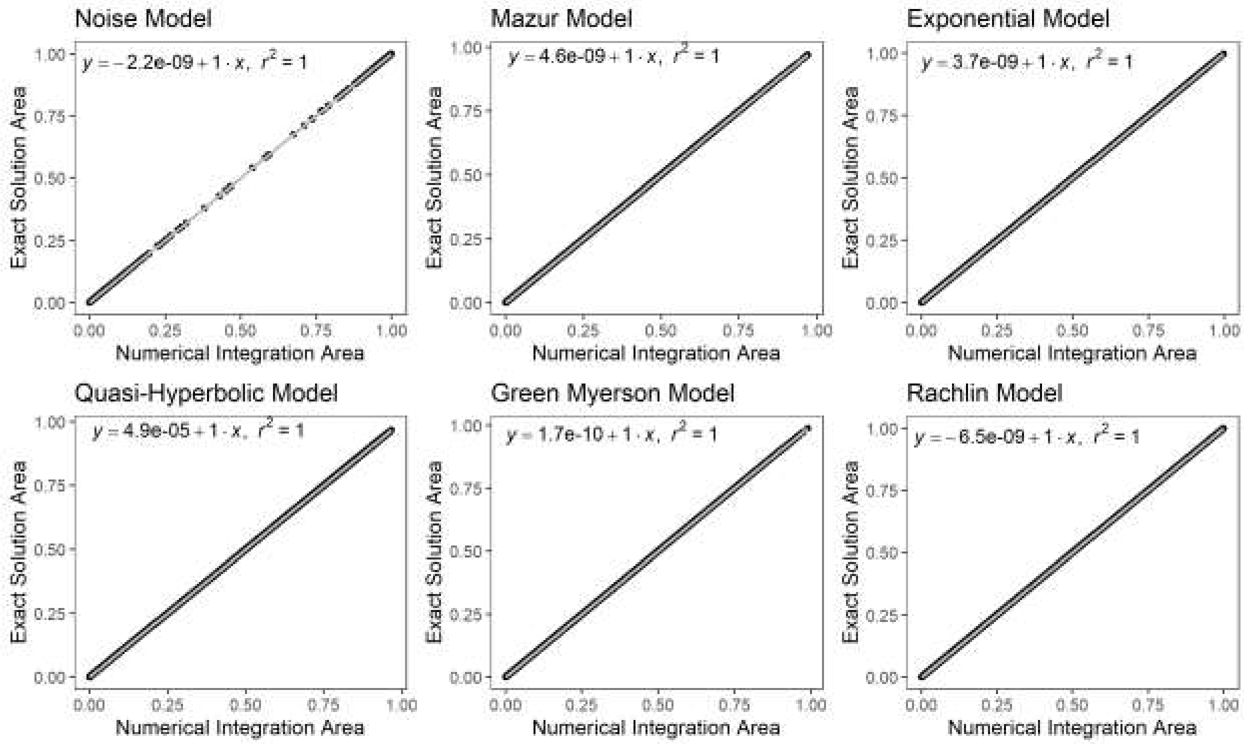
These plots illustrate how modeled discounting processes can be used to focus on specific delays of interests—even if not directly assessed. Using a common delay range, MB-AUC can be performed to provide an area proportion that can be compared across modeled (MB-AUC) and empirical (PB-AUC) analyses.

Figure 3. Relationship between Exact and Numerical Integration Area Overall



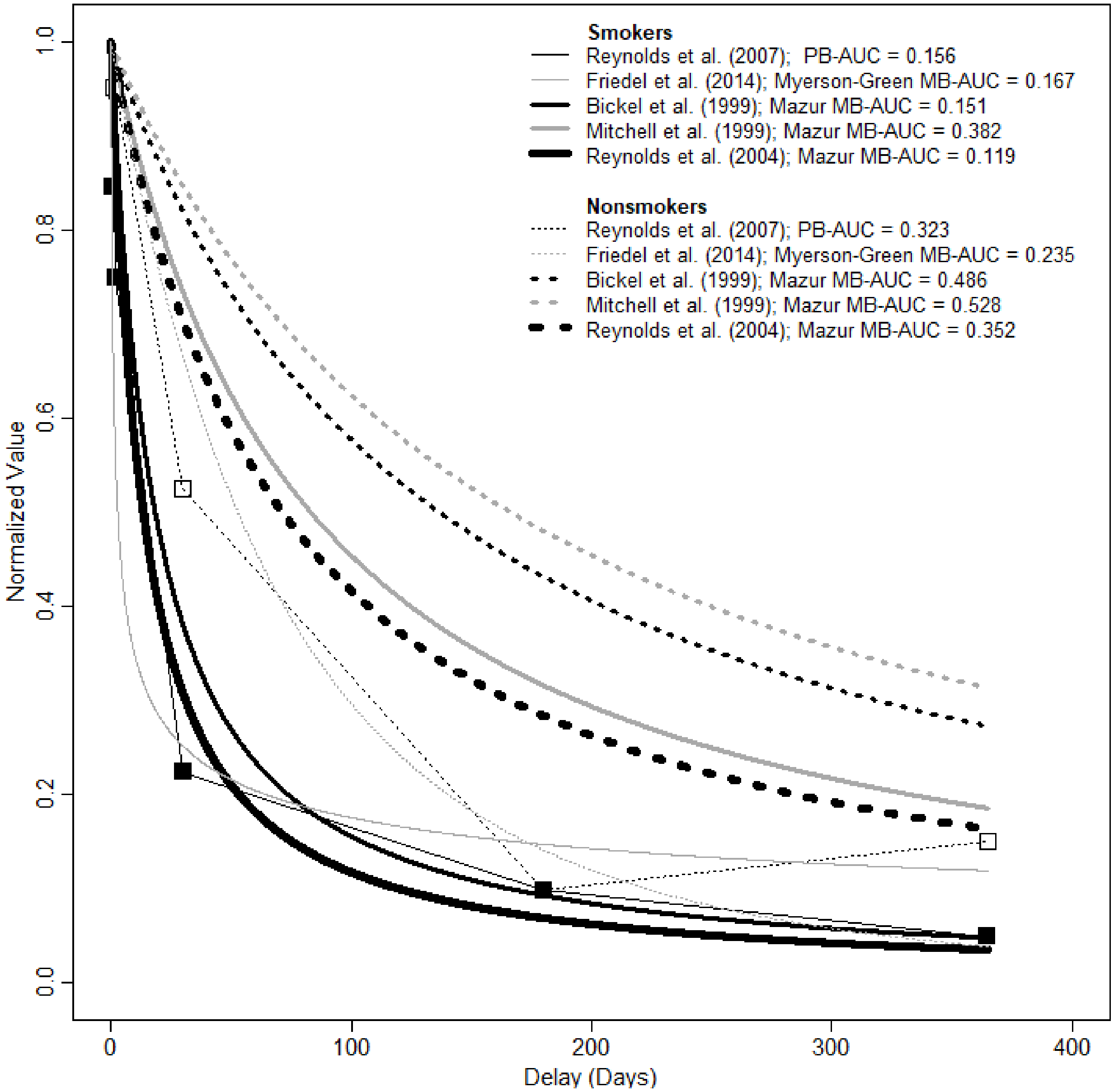
The overall relationship between the results from numerical integration and exact solution. The two methods provide essentially identical results, though the exact solution method provides MB-AUC without the need for the source data.

Figure 4. Relationship between Exact and Numerical Integration Area by Model



These plots depict the relationship between numerical integration and exact solutions for calculating MB-AUC. Both methods are specific to model structures and comparisons of the two approaches revealed essentially identical results.

Figure 5. Combined PB-AUC and MB-AUC Delay Discounting Results



This plot illustrates the relationship between PB-AUC and MB-AUC when MB-AUC is adjusted to reflect the delays from the PB-AUC. Sharing the same delay range, the AUC of non-smoking controls in five published studies was generally greater than clinical populations even when the original study analyses differed.